

Integrals

Start with dx -- this means "a little bit of x " or
"*a little change in x* "

If we add up a whole bunch of little changes in x , we
get the "*total change of x* " --

**A tautology question: If you add up all the *changes*
in x as x changes from 2 to 7, what do you get?**

A. 0

B. 2

C. 5

D. 7

E. cannot be determined

We write this in integral notation as:

$$\int_2^7 1 dx = 5$$

If $y = f(x)$, then we write $dy = f'(x) dx$.

To add up all the "*little changes in y*" as x changes from 2 to 7, we should write $\int_2^7 f'(x) dx$ or $\int_2^7 \frac{df}{dx} dx$

... and the answer should be the total change in y

as x changes from 2 to 7, in other words

$$\int_2^7 \frac{df}{dx} dx = f(7) - f(2)$$

This is the content of the fundamental theorem of calculus!

The fundamental theorem of calculus...

gives the connection between derivatives and integrals. It says you can calculate

$$\int_a^b g(x) dx$$

precisely if you can find a function whose derivative is $g(x)$. And the result is the difference between the value of the "anti-derivative" function evaluated at b minus the same function evaluated at a .

Basic antiderivative formulas:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{except for } n = -1 \quad \int \frac{1}{x} dx = \ln(x) + C$$

$$\int \cos(x) dx = \sin(x) + C$$

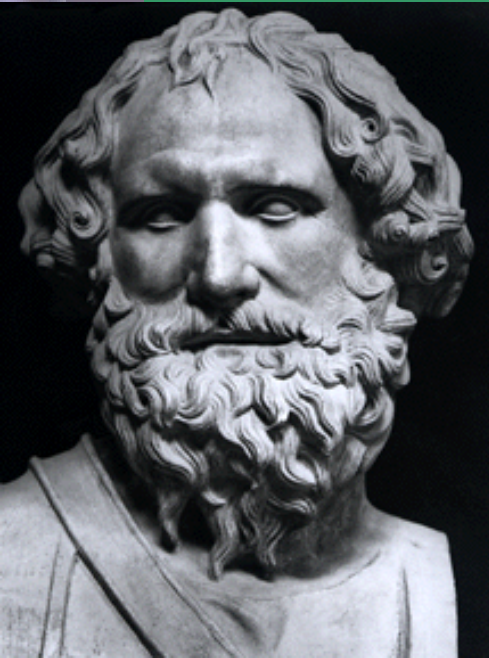
$$\int e^x dx = e^x + C$$

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C \quad \int \frac{1}{1+x^2} dx = \arctan(x) + C$$

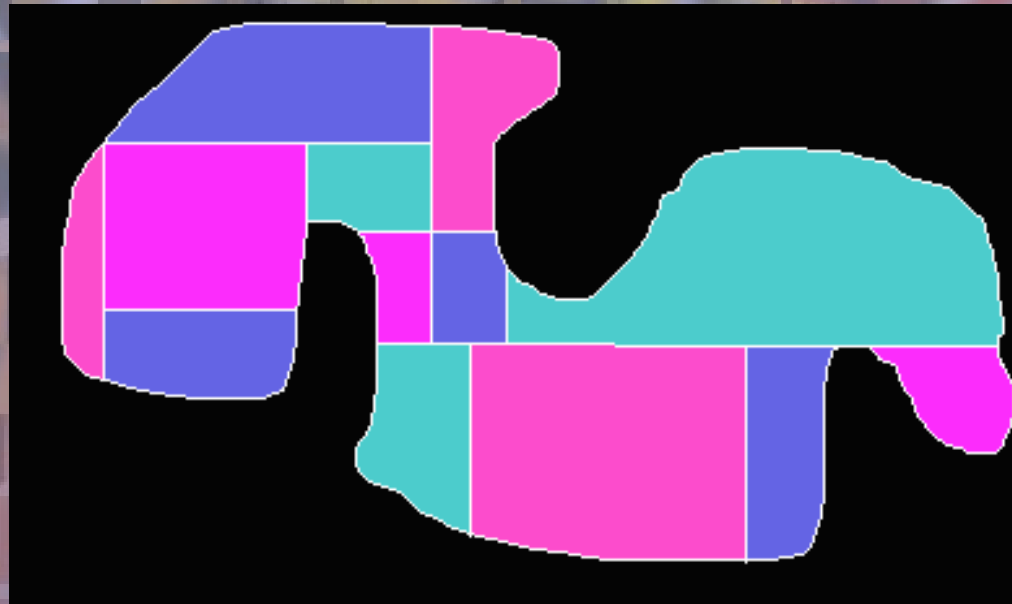
Integrals and Areas

A problem that was around long before the invention of calculus is to find the area of a general plane region (with curved sides).

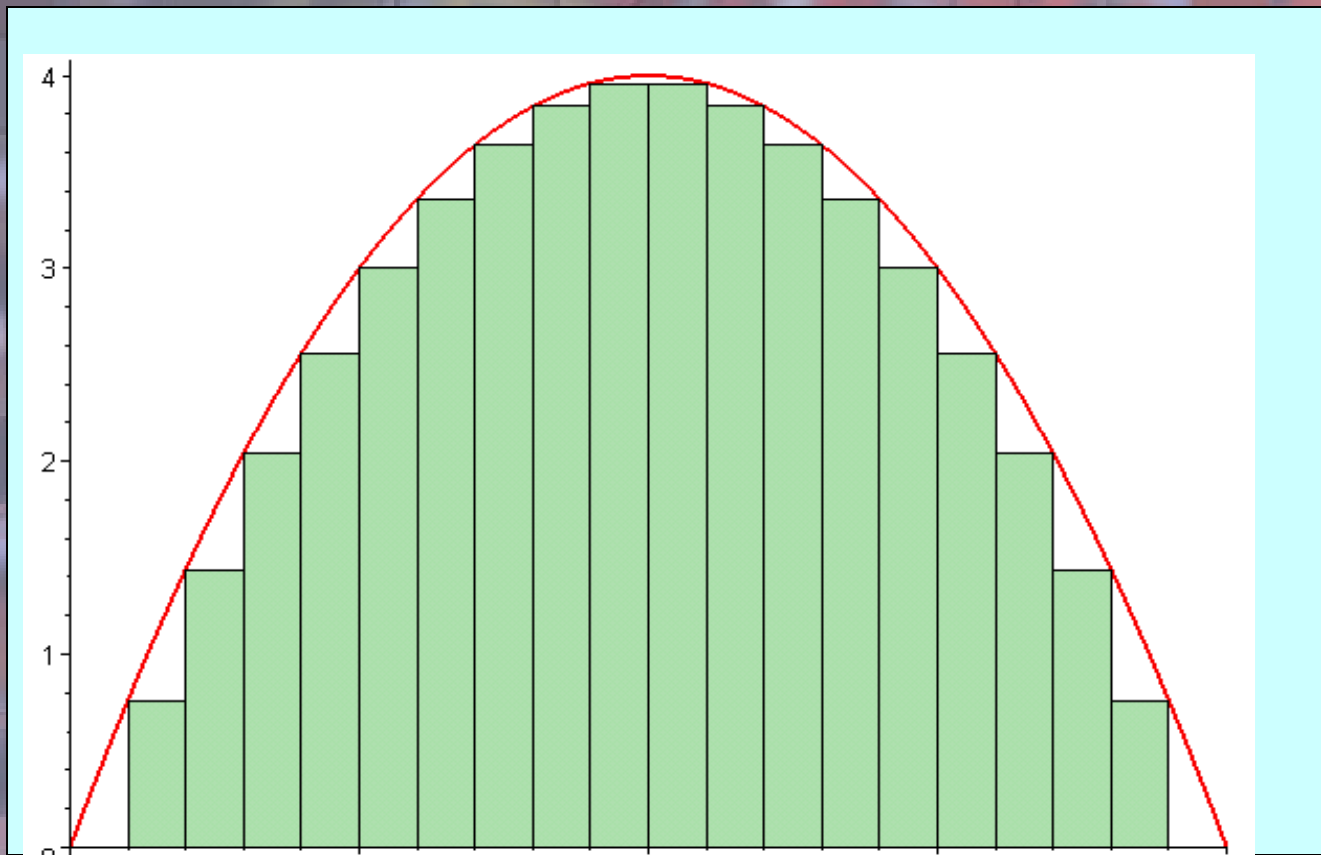


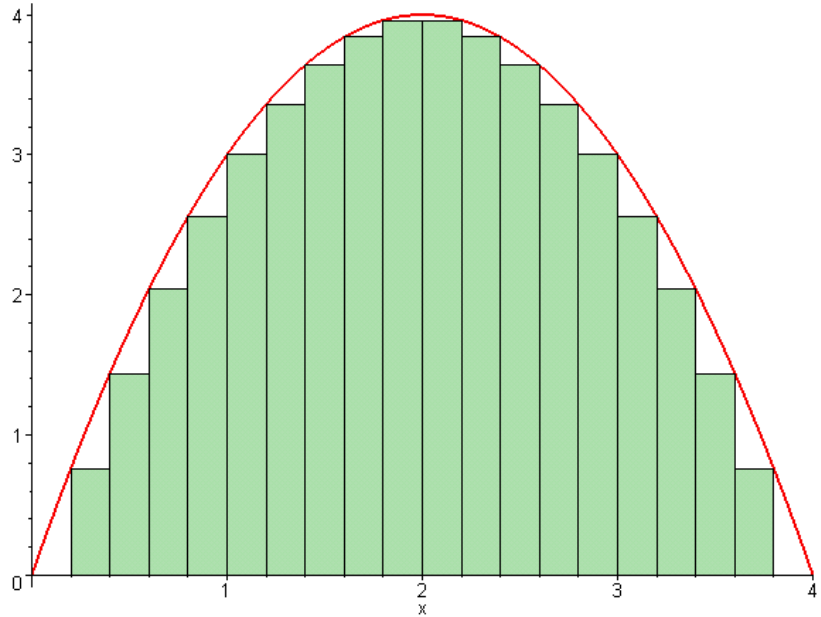
And a method of solution that goes all the way back to Archimedes is to divide the region up into lots of little regions, so that you can find the area of almost all of the little regions, and so that the total area of the ones you can't measure is very small.

By Newton's time, people realized that it would be sufficient to handle regions that had three straight sides and one curved side (or two or one straight side - the important thing is that all but one side is straight). Essentially all regions can be divided up into such regions.



These all-but-one-side-straight regions look like areas under the graphs of functions. And there is a standard strategy for calculating (at least approximately) such areas. For instance, to calculate the area between the graph of $y = 4x - x^2$ and the x axis, we draw it and subdivide it as follows:





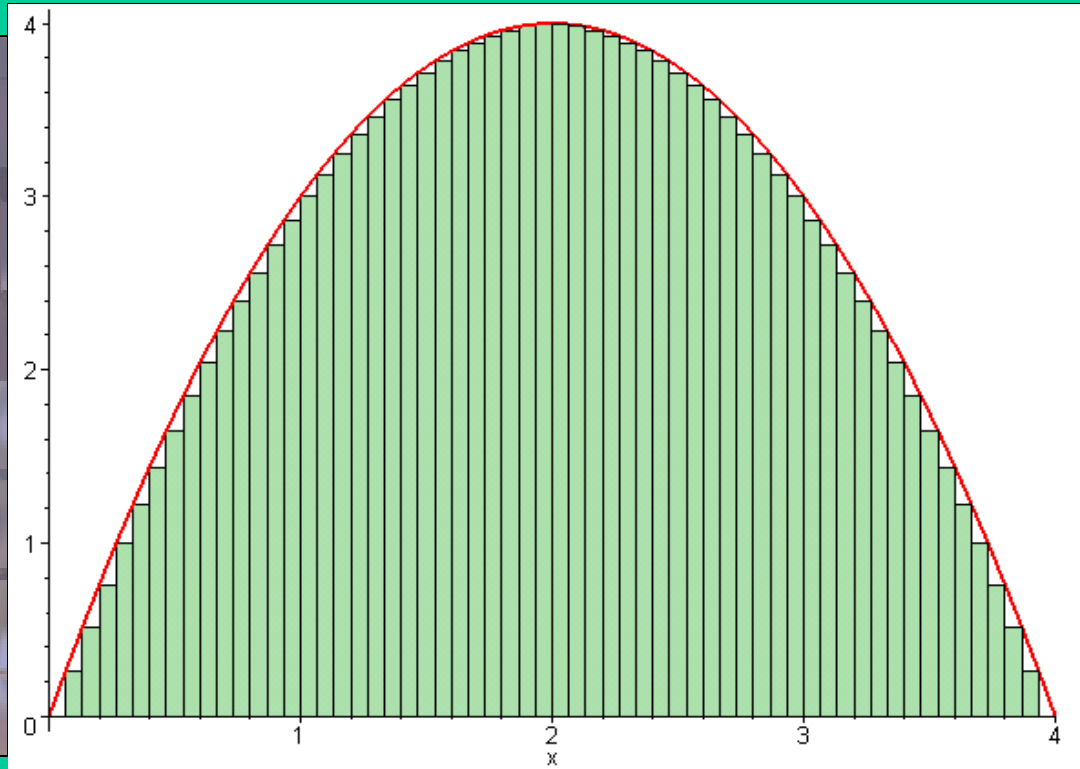
Since the green pieces are all rectangles, their areas are easy to calculate. The blue parts under the curve are relatively small, so if we add up the areas of the rectangles, we won't be far from the area under the curve. For the record, the total area of all the green rectangles is: $\frac{246}{25}$

whereas the actual area under the curve is:

$$\int_0^4 4x - x^2 dx = \frac{32}{3}$$

Also for the record, $246/25 = 9.84$ while $32/3$ is about **10.6667**.

We can improve the approximation by dividing into more rectangles:

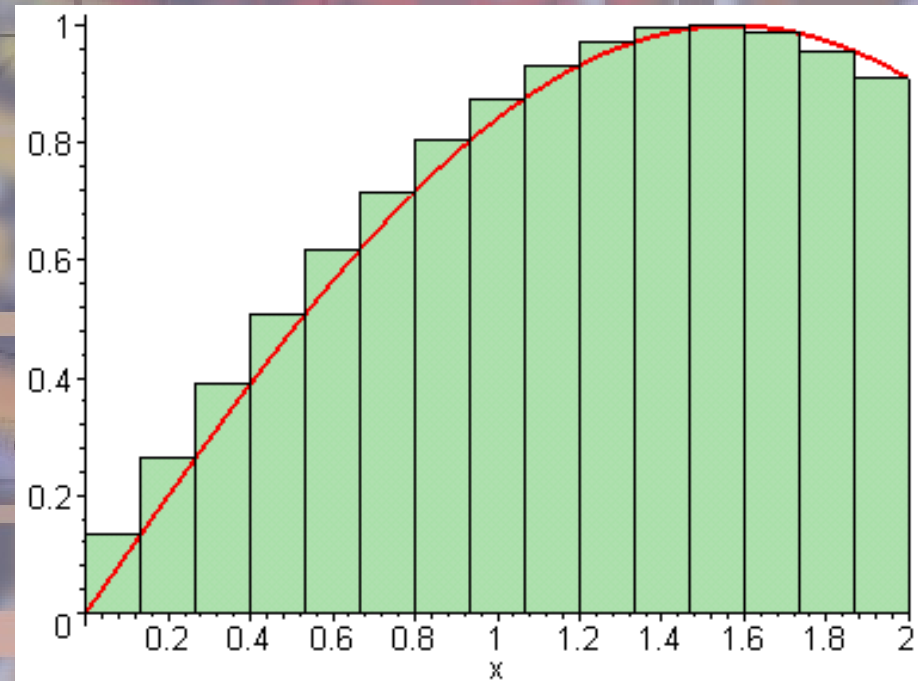


Now there are 60 boxes instead of 20, and their total area is: $\frac{7018}{675}$ which is about 10.397. Getting better. We can in fact take the limit as the number of rectangles goes to infinity, which will give the same value as the integral. *This was Newton's and Leibniz's great discovery -- derivatives and integrals are related and they are related to the area problem.*

Green graph

Again, recall that one way to look at integrals is as areas under graphs, and we approximate these areas as sums of areas of rectangles.

This is a picture of the “right endpoint” approximation to the integral of a function.



If we are approximating the integral of $f(x)$ over the interval from a to b using n rectangles, then each rectangle has width $\frac{b-a}{n}$. The right side of the i th rectangle is at

$x = a + \frac{i(b-a)}{n}$, and so the area of the i th rectangle is

$f\left(a + \frac{i(b-a)}{n}\right) \frac{b-a}{n}$. The sum of the areas of the rectangles is

thus $\sum_{i=1}^n \frac{(b-a)f\left(a + \frac{i(b-a)}{n}\right)}{n}$, and the limit of this sum as n

approaches infinity is the integral $\int_a^b f(x)dx$. An example will help \longrightarrow

First, we need a $1/n$ for our $(b - a)/n$.

So we can rewrite the expression under

the summation sign as $(\frac{1}{n})(\frac{i}{n})^3$. Now we

need to figure out a and b . For $(b - a)/n = 1/n$,
we need $b - a = 1$. And i/n appears in the other

factor, we should choose $a = 0$. If $f(x) = x^3$,

then we'll have $\frac{(b-a)f(a+\frac{i(b-a)}{n})}{n} = \frac{1}{n} (\frac{i}{n})^3$. Thus the limit

of the sum is equal to $\int_0^1 x^3 dx = \frac{1}{4}$.

Area between two curves:

A standard kind of problem is to find the area above one curve and below another (or to the left of one curve and to the right of another). This is easy using integrals.

Note that the "**area between a curve and *the axis***" is a special case of this problem where one of the curves simply has the equation $y = 0$ (or perhaps $x = 0$)

Solving area problems

1. Graph the equations if possible

**2. Find points of intersection of the curves to
limits of integration, if none are given**

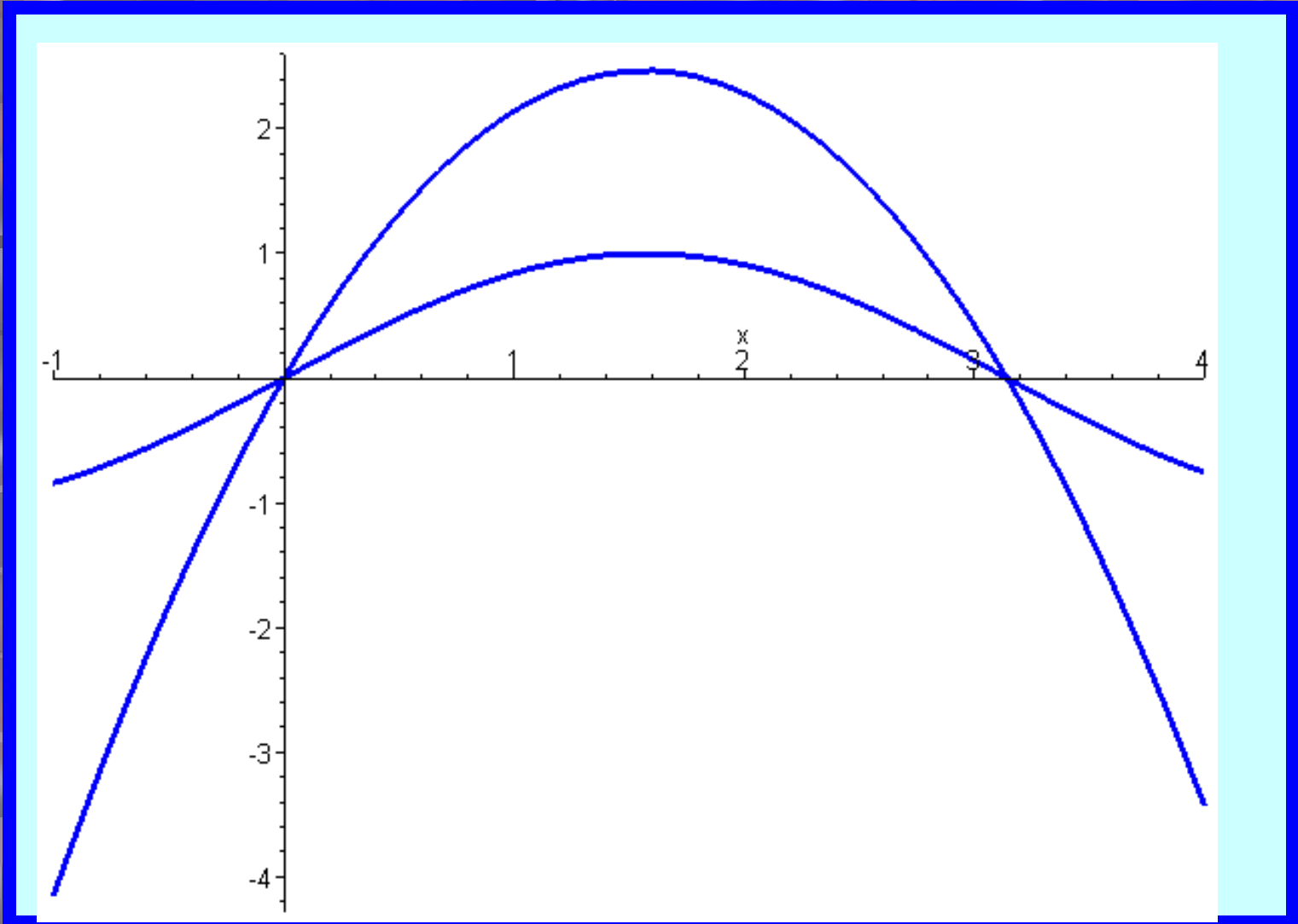
determine

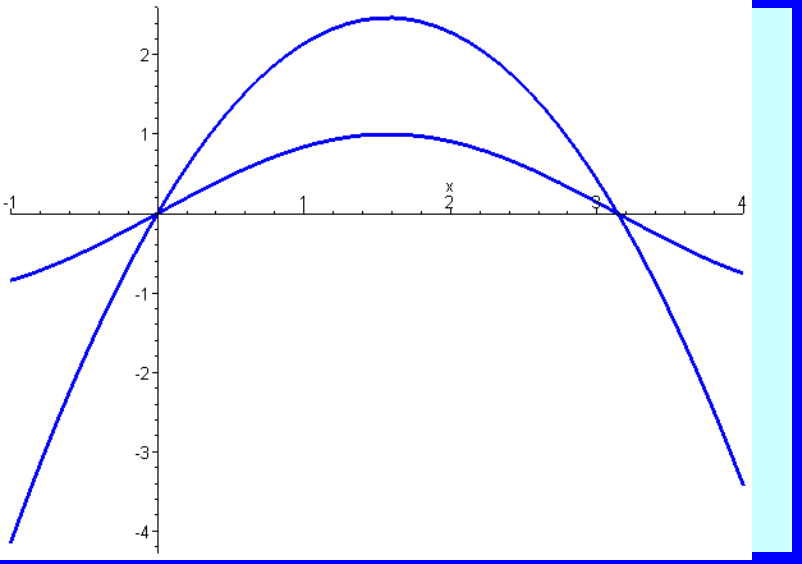
**3. Integrate the top curve's function minus the
curve's (or right curve minus left curve).**

bottom

Example:

Find the area between the graphs of $y=\sin(x)$ and $y=x(\pi-x)$





It's easy to see that the curves intersect on the x-axis, and the values of x are 0 and π .

The parabola is on top, so we integrate:

$$\int_0^{\pi} x(\pi - x) - \sin(x) \, dx = \frac{\pi^3}{6} - 2$$

And this is the area between the two curves.

Position, velocity, and acceleration:

Since velocity is the derivative of position and acceleration is the derivative of velocity,

Velocity is the integral of acceleration, and position is the integral of velocity.

(Of course, you must know *starting values of position and/or velocity* to determine the constant of integration.)



Example...

An object moves in a force field so that its acceleration at time t is $a(t) = t^2 - t + 12$ (meters per second squared). Assuming the object is moving at a speed of 5 meters per second at time $t=0$, determine how far it travels in the first 10 seconds.

Solution...

First we determine the velocity, by integrating the acceleration. Because $v(0) = 5$, we can write the velocity $v(t)$ as $5 +$ a *definite* integral, as follows:

$$v(t) = 5 + \int_0^t a(\tau) d\tau = 5 + \int_0^t \tau^2 - \tau + 12 d\tau = 5 + \frac{t^3}{3} - \frac{t^2}{2} + 12t$$

The distance the object moves in the first 10 seconds is the total change in position. In other words, it is the integral of dx as t goes from 0 to 10. But $dx = v(t) dt$. So we can write:

$$\begin{aligned} \text{(distance traveled between } t=0 \text{ and } t=10) &= \int_0^{10} v(t) dt \\ &= \int_0^{10} 5 + \frac{t^3}{3} - \frac{t^2}{2} + 12t dt = 3950/3 = 1316.666... \text{ meters .} \end{aligned}$$

Methods of integration

Before we get too involved with applications of the integral, we have to make sure we're good at calculating antiderivatives.

There are four basic tricks that you have to learn (and hundreds of *ad hoc* ones that only work in special situations):

- 1. Integration by substitution (chain rule in reverse)**
- 2. Trigonometric substitutions (using trig identities to your advantage)**
- 3. Partial fractions (an algebraic trick that is good for more than doing integrals)**
- 4. Integration by parts (the product rule in reverse)**

We'll do #1 this week, and the others later. LOTS of practice is needed to master these!

Substitution

In some ways, substitution is the most important technique, because *every* integral can be worked this way (at least in theory).

The idea is to remember the chain rule: If G is a function of u and u is a function of x , then the derivative of G with respect to x is:

$$\frac{dG}{dx} = G'(u) u'(x)$$

For instance...

e^{x^2} could be considered as e^u where $u = x^2$.

To differentiate e^{x^2} then, we use that the derivative of e^u is e^u :

$$\frac{d}{dx} e^{x^2} = \frac{d}{du} (e^u) \frac{d}{dx} (x^2) = e^u (2x) = 2xe^{x^2}$$

Now we'll turn this around...

To do an integral problem...

For a problem like $\int x^3 e^{x^4} dx$

we suspect that the x^4 should be considered as u
and then $x^3 dx$ is equal to $du/4$.

And so:

$$\int x^3 e^{x^4} dx = \int e^u \frac{du}{4} = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C$$

In general...

In substitution, you

1. Separate the integrand into factors
2. Figure out which factor is the most complicated
3. Ask whether the *other* factors are the derivative of some (compositional) part of the complicated one.

This provides the clue as to what to set u equal to.

Here's another one:

$\int \frac{x}{(2x^2 + 5)^3} dx$ -- the complicated factor is clearly **the denominator** (partly by virtue of being in the denominator!)

and the rest (**x dx**) is a constant times the differential of x^2 -- but it's a good idea to try and make u substitute for as much of the complicated factor as possible.

And if you think about it, **x dx** is a constant times the differential of $2x^2+5$! So we let $u = 2x^2+5$, then $du = 4 x dx$, in other words **x dx = du / 4** . So we can substitute:

$$\int \frac{x}{(2x^2 + 5)^3} dx = \frac{1}{4} \int \frac{1}{u^3} du = \frac{1}{4} \frac{u^{-2}}{-2} + C = -\frac{1}{8(2x^2 + 5)^2} + C$$

Now you try a couple...

$$\int_0^{\sqrt{\pi/2}} x \cos x^2 dx =$$

A) 0

B) 1/2

C) 1

D) $\pi/2$

E) $\sqrt{\pi}$

Find

$$\int_0^{\pi/4} \sec^2 x \sin(\tan x) dx$$

A) $\pi/2$

E) $\pi/2 - \sin 1$

B) $1 - \pi/4$

F) $\pi/4 + \cos 1$

C) $\sin 1$

G) $1 + 3\pi/4$

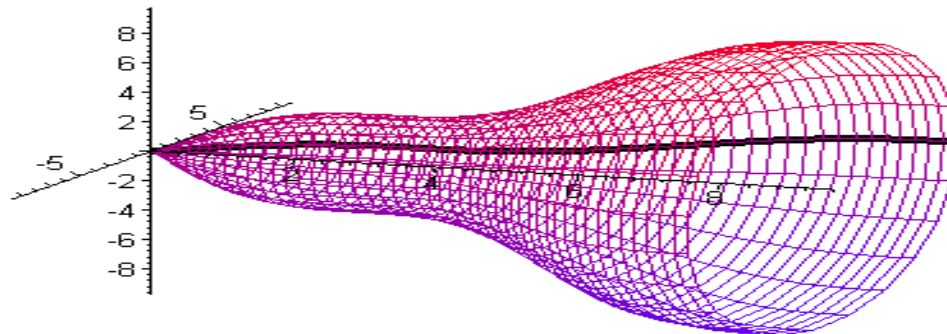
D) $1 - \cos 1$

H) $1 + \tan 1$

Surfaces of revolution: Volume

A "**surface of revolution**" is formed when a curve is revolved around a line (usually the x or y axis). The curve sweeps out a surface

Interesting problems that can be solved by integration are to find the volume enclosed inside such a surface or to find its surface area.

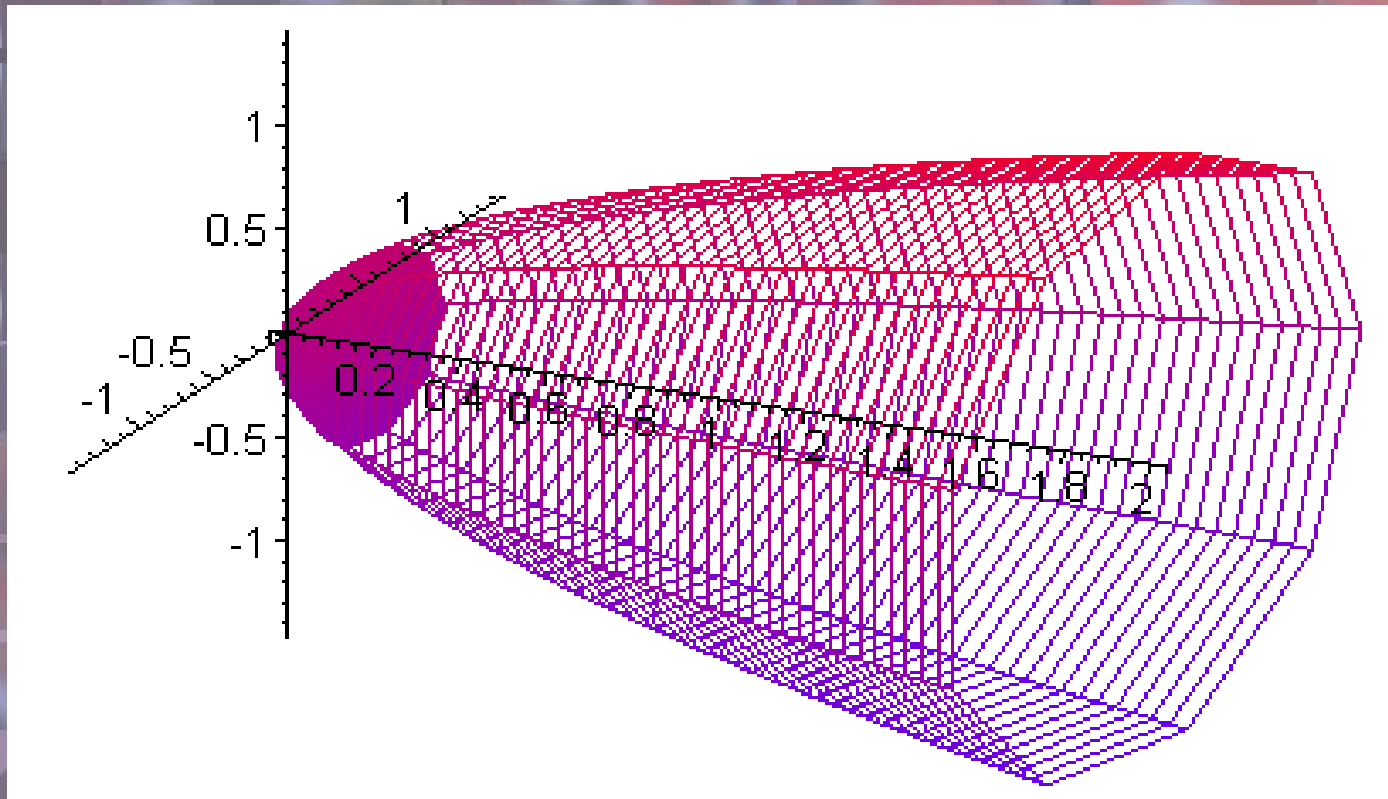


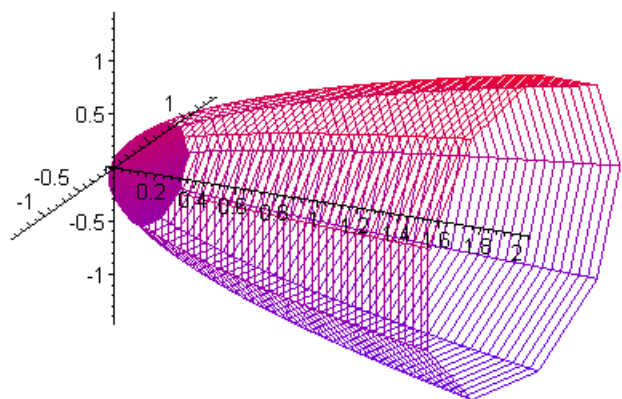
Volumes

You might already be familiar with finding volumes of revolution.

Once a surface is formed by rotating around the x -axis, you can sweep out the volume it encloses with disks perpendicular to the x axis.

Here is the surface formed by revolving $y = \sqrt{x}$ around the x axis for x between 0 and 2, showing one of the disks that sweep out the volume:





To calculate the volume enclosed inside the surface, we need to add up the volumes of all the disks.

The disks are (approximately) cylinders turned sideways, and the disk centered at $(x,0)$ has radius \sqrt{x} and width (or height) dx . The volume of the disk is thus

$$\pi(\sqrt{x})^2 dx, \text{ or } \pi x dx$$

To find the total volume of the solid we have to integrate this quantity for x from 0 to 2. We get

$$V = \int_0^2 \pi x dx = 2\pi \quad \text{cubic units}$$

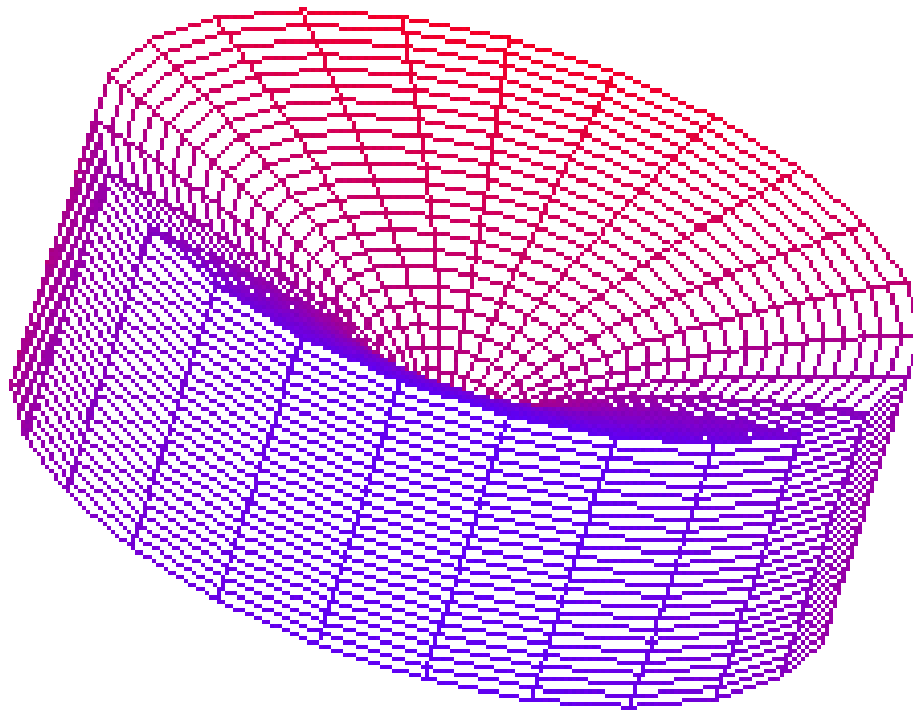
A formula for volume:

In general, if the piece of the graph of the function of $y = f(x)$ between $x = a$ and $x = b$ is revolved around the x axis, the volume inside the resulting solid of revolution is calculated as:

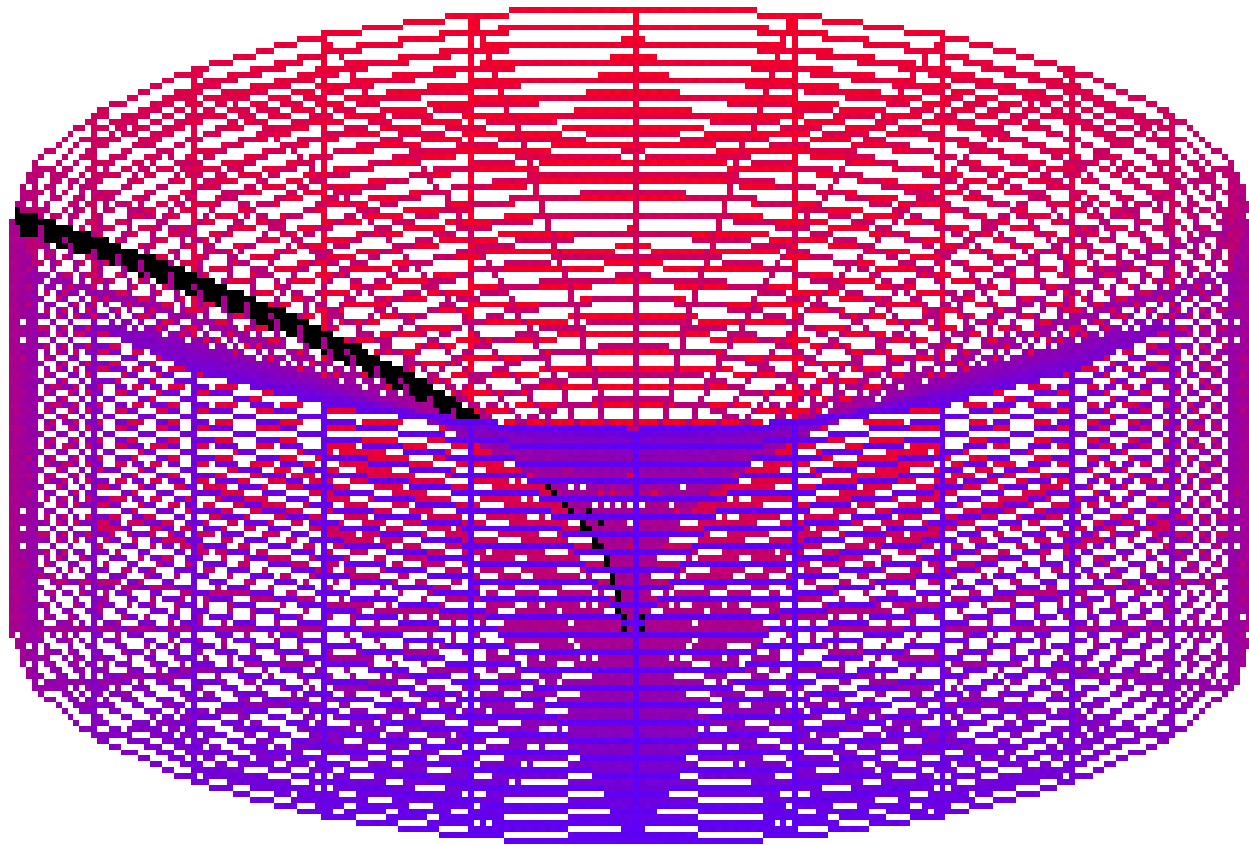
$$V = \int_a^b \pi (f(x))^2 dx$$

The same sort of formula applies if we rotate the region between the y -axis and a curve around the y -axis (just change all the x 's to y 's).

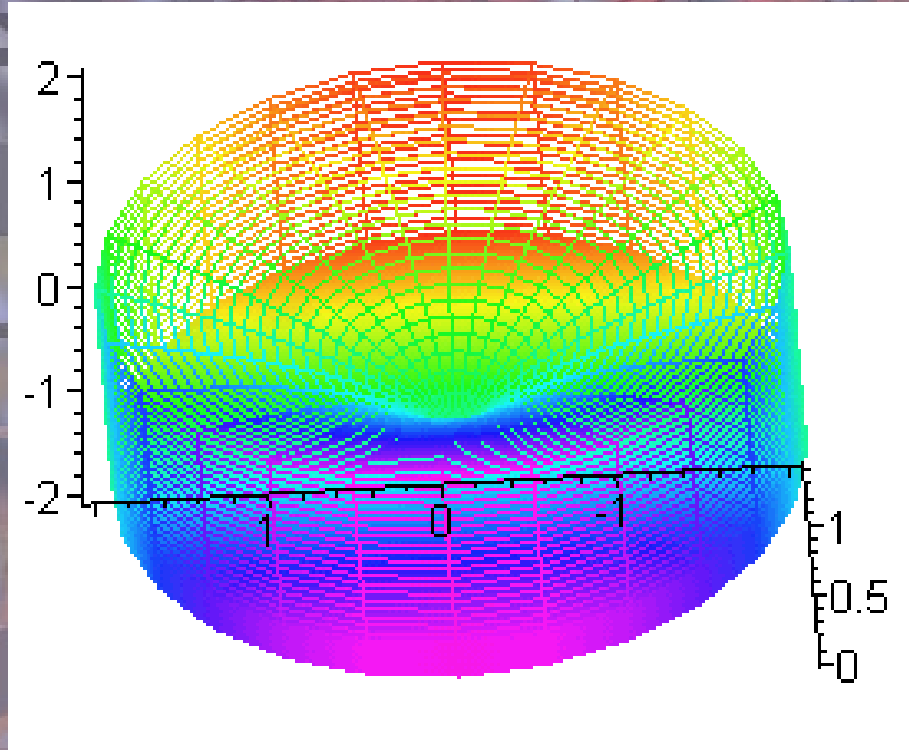
A different kind of problem is to rotate the region between a curve and the x axis around the y axis (or vice versa). For instance, let's look at the same region (between $y=0$ and $y=\sqrt{x}$ for x between 0 and 2), but rotated around the y axis instead:



Here is the surface being swept out by the generating curve:



We could sweep out this volume with “washers” with inner radius y^2 and outer radius 2 as y goes from 0 to $\sqrt{2}$



Each washer is (approximately) a cylinder with a hole in the middle. The volume of such a washer is then the volume of the big cylinder minus the volume of the hole.

The volume of the washers...

For the washer centered at the point $(0, y)$, the radius of the outside cylinder is always equal to 2, and the radius of the hole is equal to the corresponding x (which, since $y = \sqrt{x}$, is equal to y^2). And the height of the washer is equal to dy . So the volume of the washer is

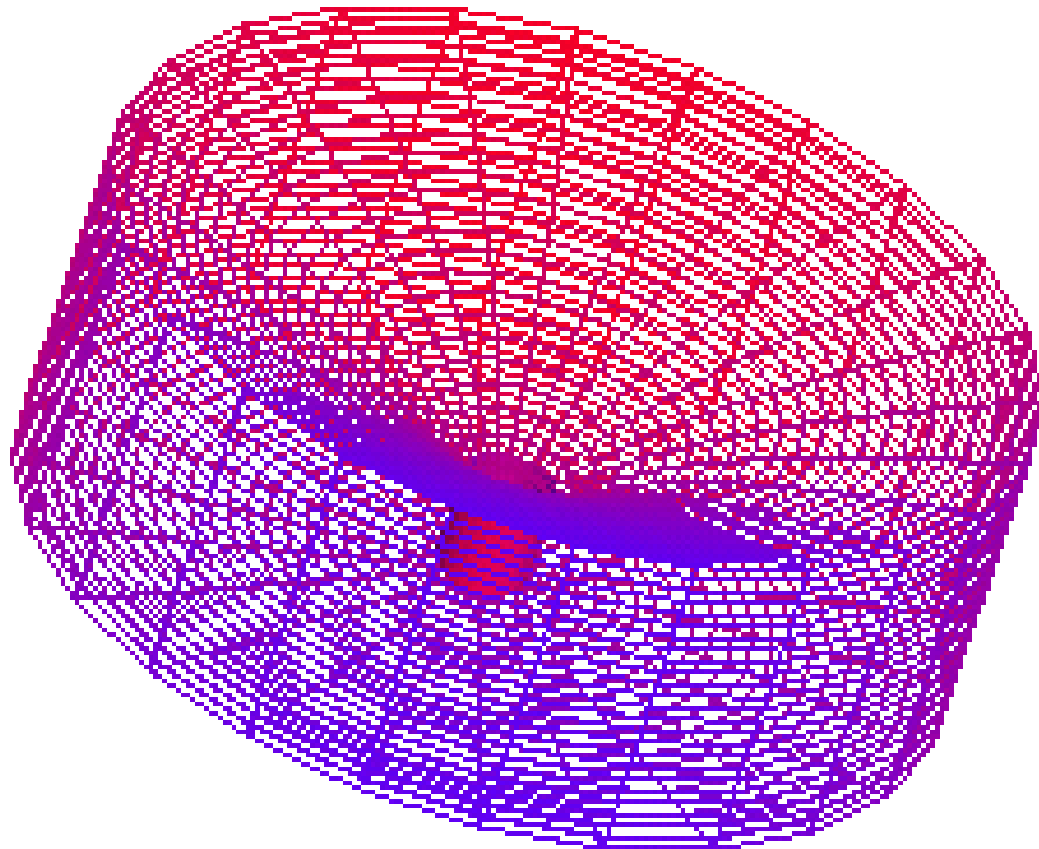
$$dV = \pi \left(2^2 - (y^2)^2 \right) dy = \left(4\pi - \pi y^4 \right) dy$$

Therefore the volume of the
entire solid is

$$V = \int_0^{\sqrt{2}} 4\pi - \pi y^4 dy = \frac{16\pi\sqrt{2}}{5}$$

cubic units

Another way to sweep out this volume is with
"cylindrical shells".



The volume of a cylindrical shell

Each cylindrical shell, if you cut it along a vertical line, can be

laid out as a rectangular box, with length $2\pi x$, with width

$y = \sqrt{x}$ and with thickness dx . The volume of the cylindrical

shell that goes through the point $(x,0)$ is thus $dV = 2\pi x\sqrt{x} dx$

So, we can calculate the volume of the entire solid to be:

$$V = \int_0^2 2\pi x^{3/2} dx = \frac{16\pi\sqrt{2}}{5}$$

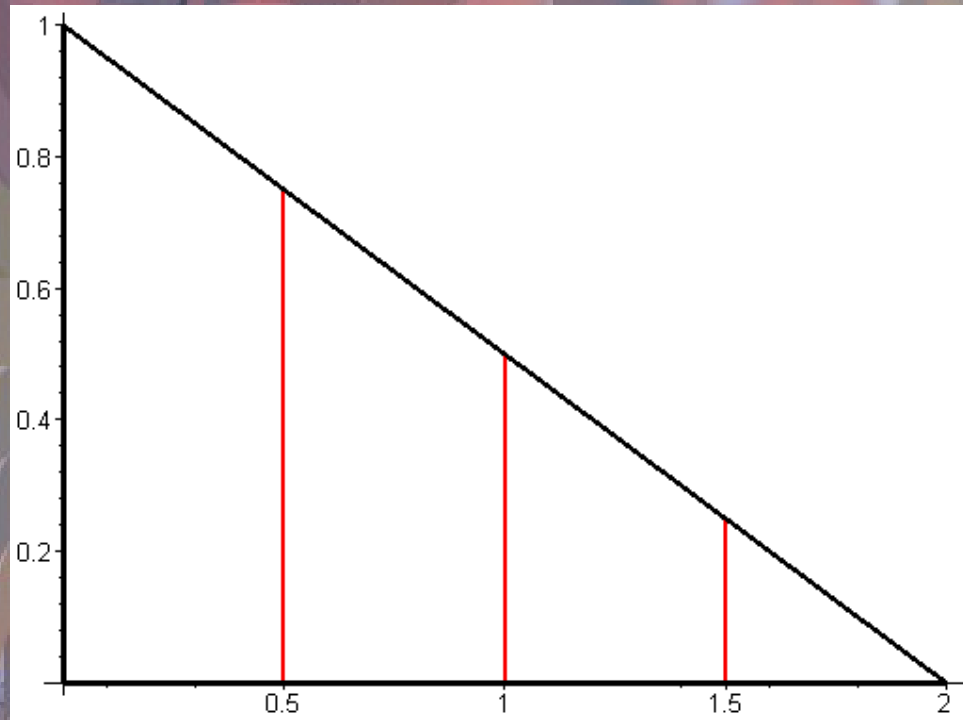
cubic units, which agrees with the answer we got the other way.

Other volumes with known cross sections

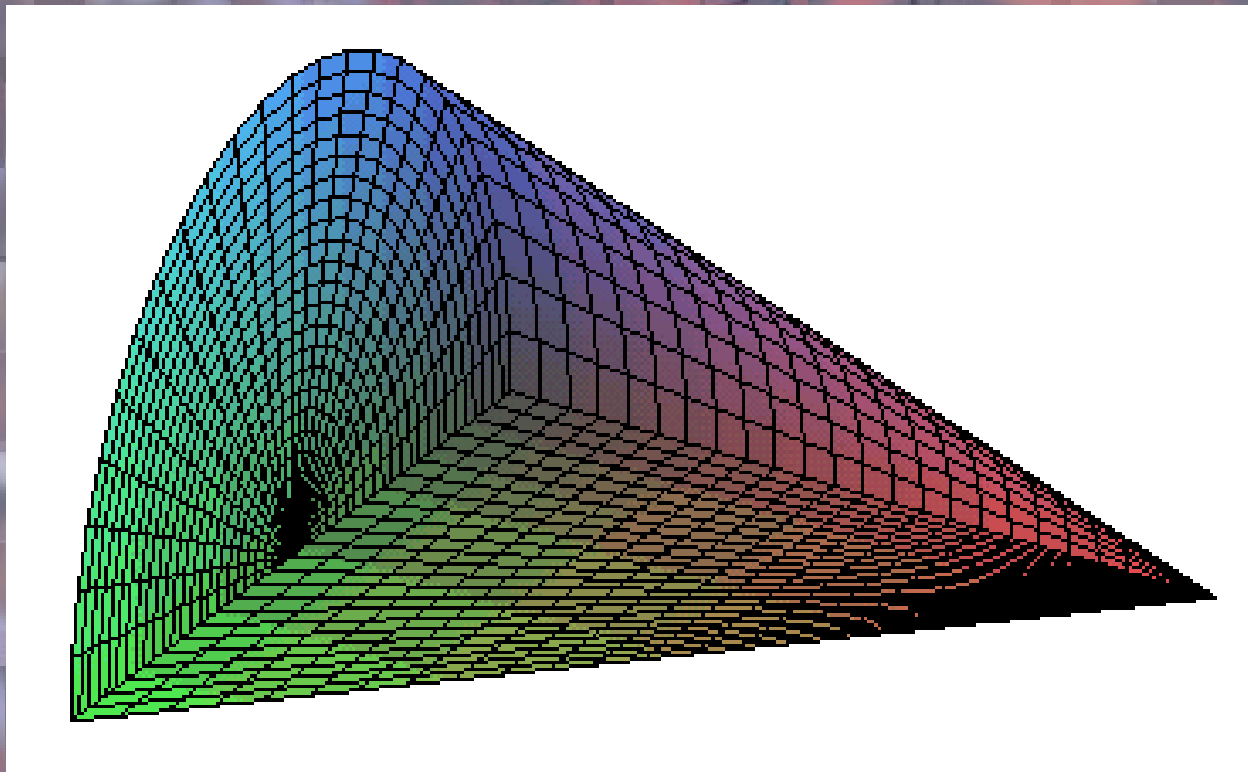
Another family of volume problems involves volumes of three-dimensional objects whose cross-sections in some direction all have the same shape.

For example: Calculate the volume of the solid S if the base of S is the triangular region with vertices $(0,0)$, $(2,0)$ and $(0,1)$ and cross sections perpendicular to the x -axis are semicircles.

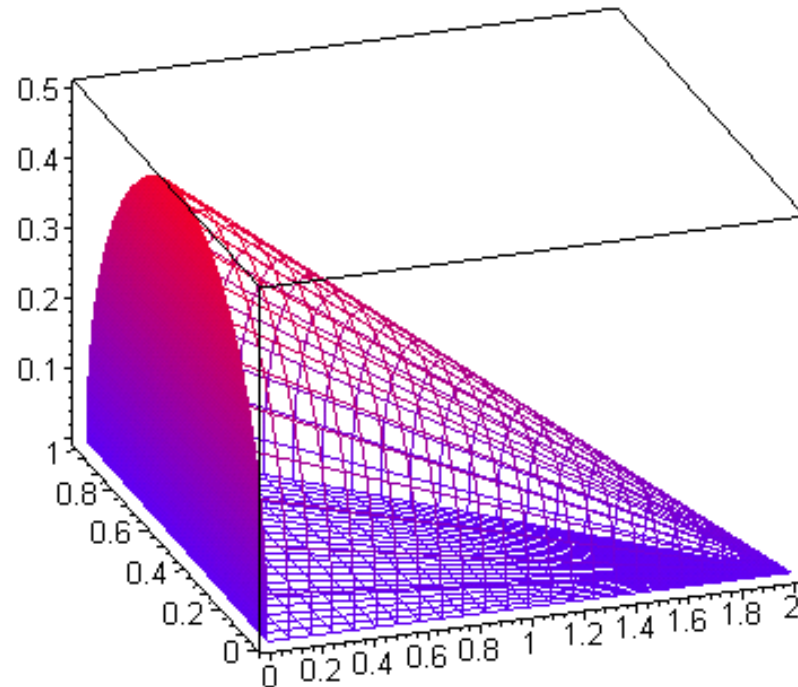
First, we have to visualize the solid. Here is the base triangle, with a few vertical lines drawn on it (perpendicular to the x -axis). These will be diameters of the semicircles in the solid.



Now, we'll make the three-dimensional plot that has this triangle as the base and the semi-circular cross sections.



From that point of view you can see some of the base as well as the cross section. We'll sweep out the volume with slices perpendicular to the x-axis, each will look like half a disk:



The volume of that object:

Since the line connecting the two points (0,1) and (2,0) has equation $y = 1 - x/2$, the centers of the half-disks are at the points $(x, 1/2 - x/4)$, and their radii are likewise $1/2 - x/4$. Therefore the little bit of volume at x is half the volume of a cylinder of radius $1/2 - x/4$ and height dx , namely

$$dV = \frac{\pi}{2} \left(\frac{1}{2} - \frac{1}{4} x \right)^2 dx$$

Therefore, the volume of the solid S is:

$$V = \frac{\pi}{2} \int_0^2 \left(\frac{1}{2} - \frac{1}{4} x \right)^2 dx = \frac{\pi}{12}$$

Finally...

Note that we could also have calculated the volume by noticing that the solid S is half of a (skewed) cone of height 2 with base radius = $1/2$.

Using the formula $V = \frac{1}{3} \pi r^2 h$ for a cone, we arrive

at the same answer, $\frac{\pi}{12}$ cubic units.